Post-Quantum Public-Key Cryptography with Isogenies

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(PhD student 2020 – 2025)

Crypto & PL & Algorithms Away Day 2024

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- Current PKC schemes - hardness comes from Integer Factorization or Discrete Logarithm Problems.

Integer Factorization

Given an integer N which is the product of two primes $N = p \times q$, find p and q.

Discrete Logarithm Problem

Given a number N which is a number g to a power a, ($N = g^a$), find a.

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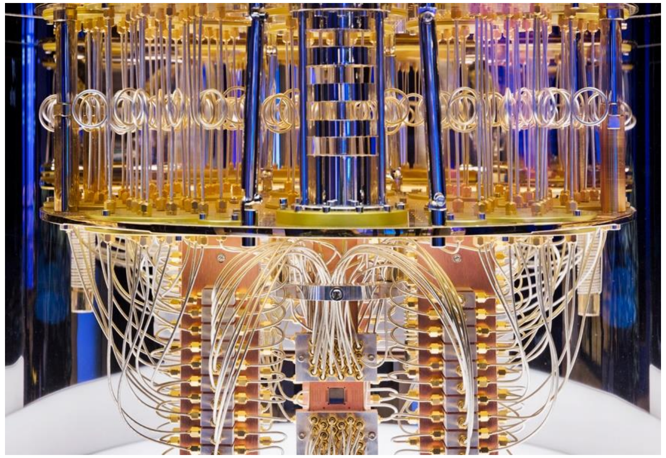

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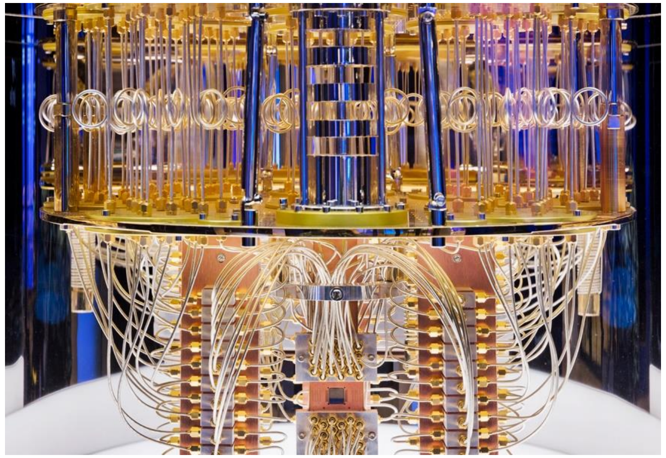

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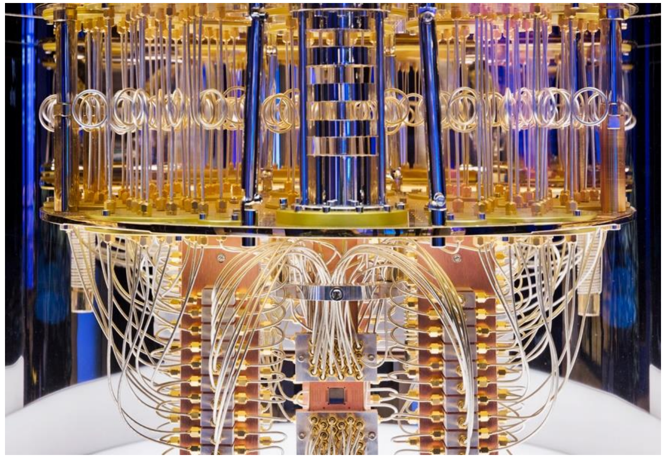

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- Post-quantum cryptography refers to new public-key schemes which rely on newer quantum resistant hard problems.

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Isogeny-based digital signature scheme **SQISign** was entered into the NIST post-quantum “competition” with the potential of being chosen to be standardized.

Aims of my research:

- Study the hardness of the underlying problems to improve confidence in isogenies, via algorithmic reductions.
- Explore new ways of applying mathematical results of quaternion algebras to isogeny-based cryptography.

An introduction to isogeny-based cryptography

Disclaimer: Using simplified, very imprecise, definitions.

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We only care about **supersingular** elliptic curves (which I won't define).

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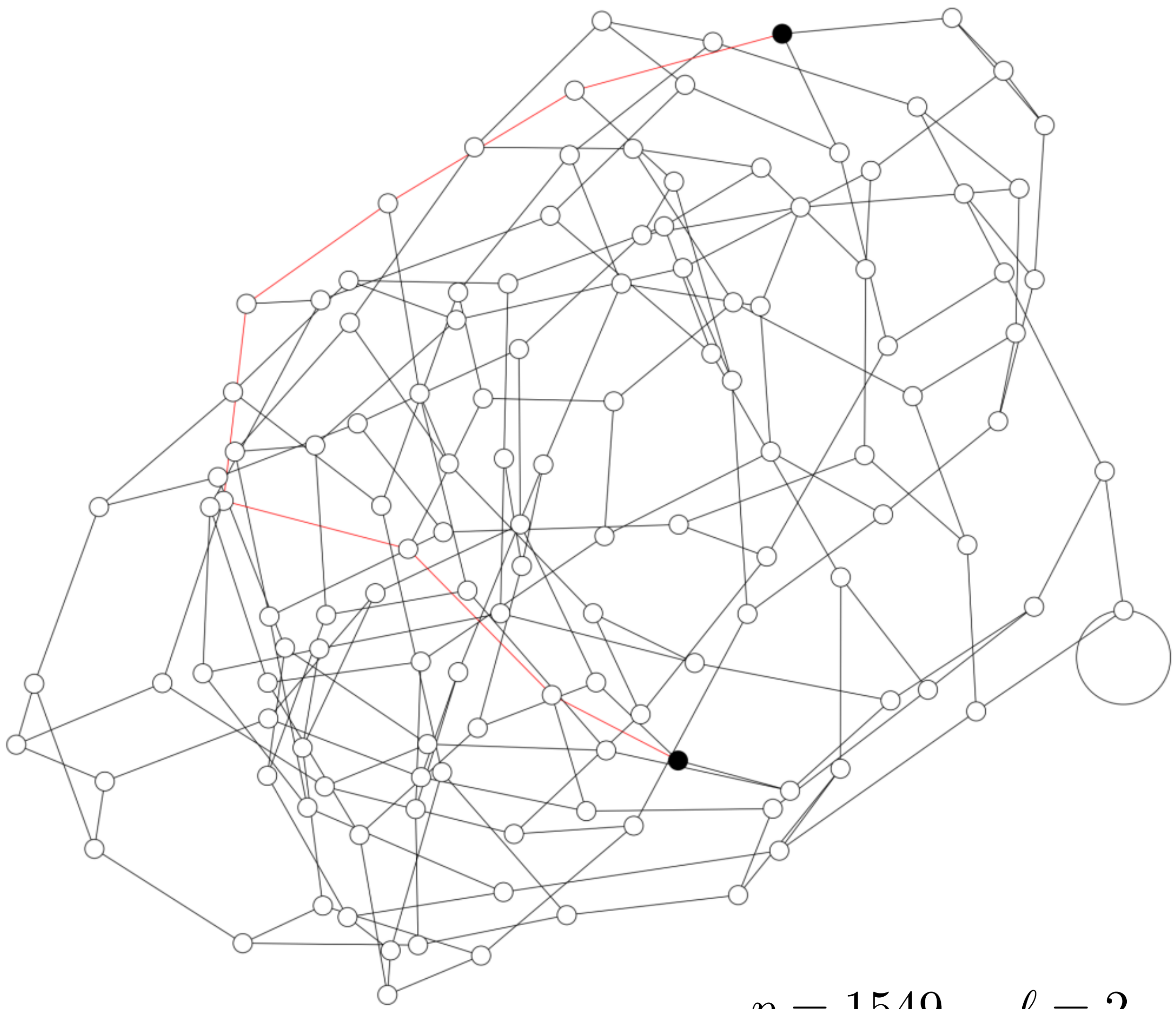

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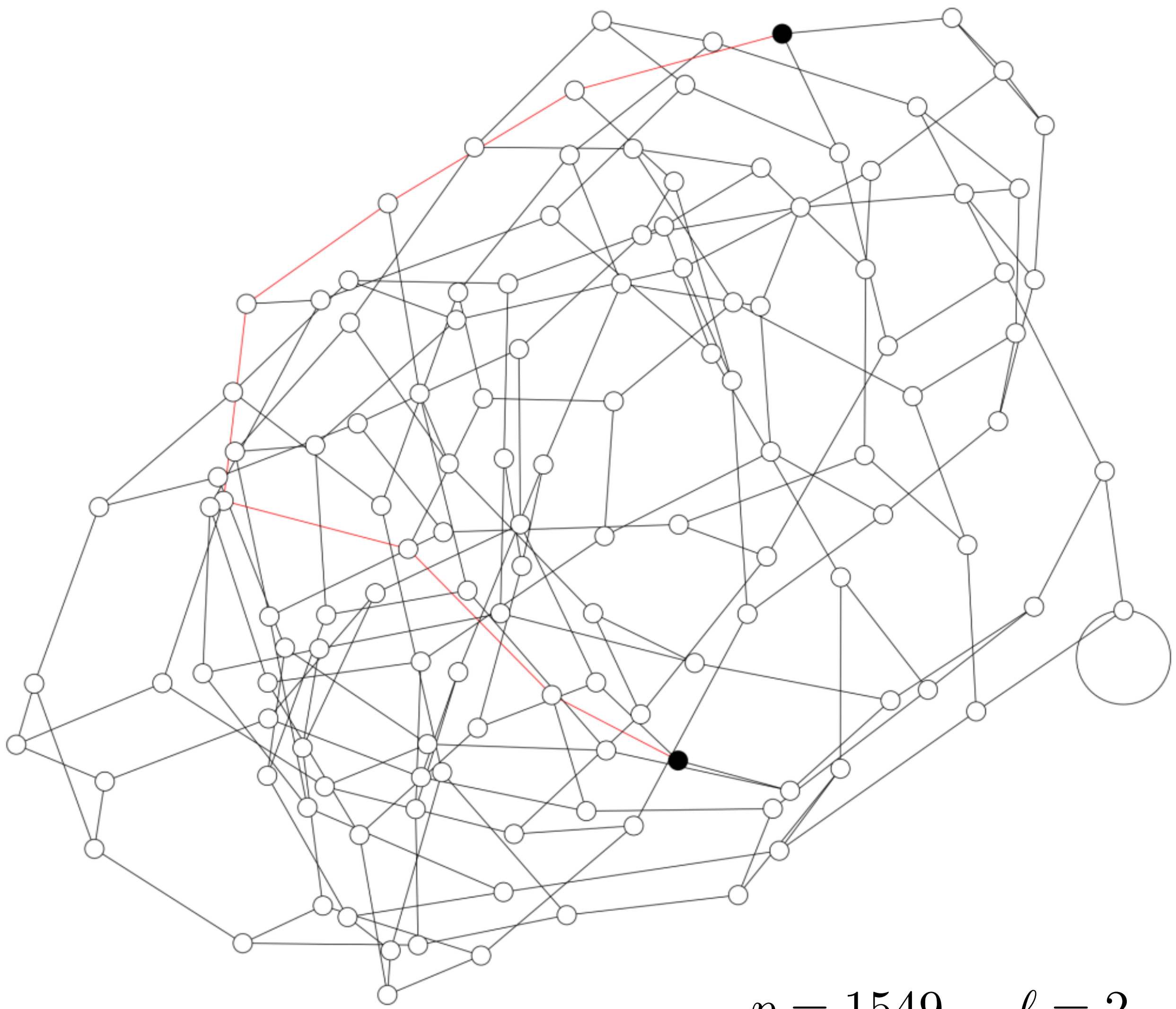
All isogeny-based cryptographic schemes walk around in these graphs.



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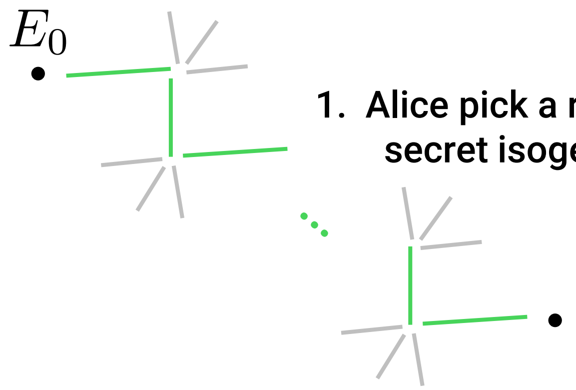

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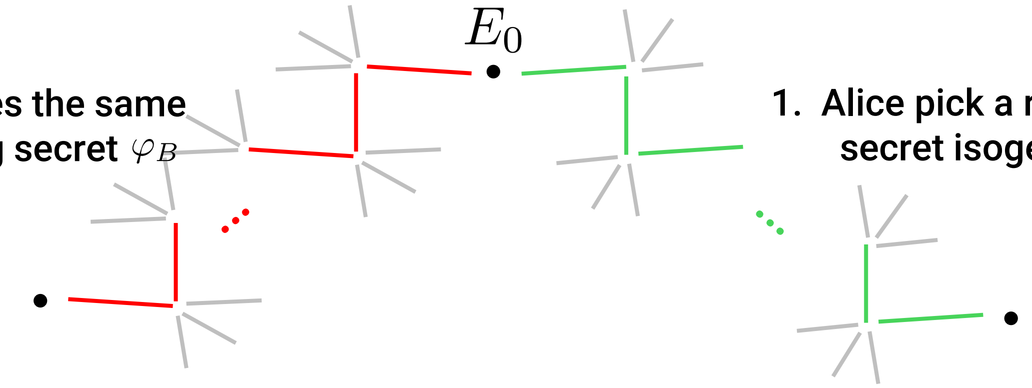

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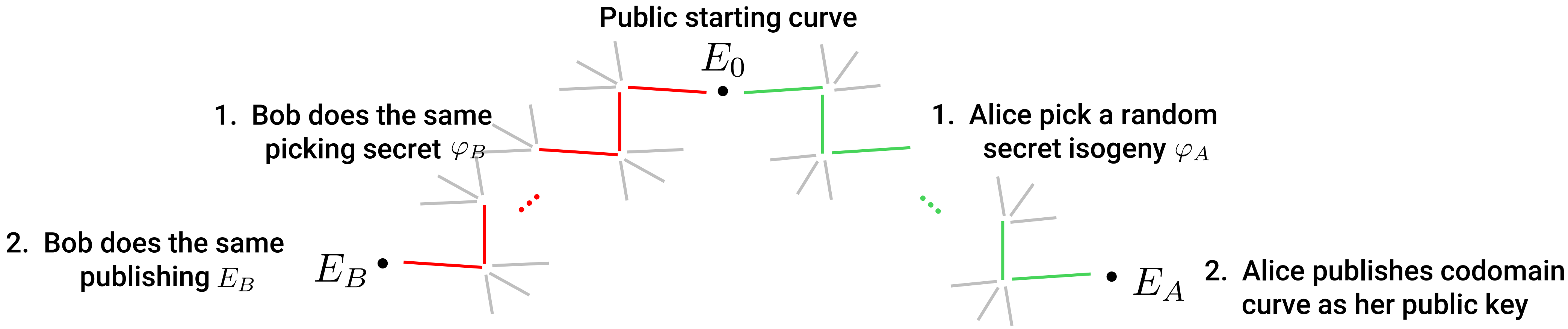

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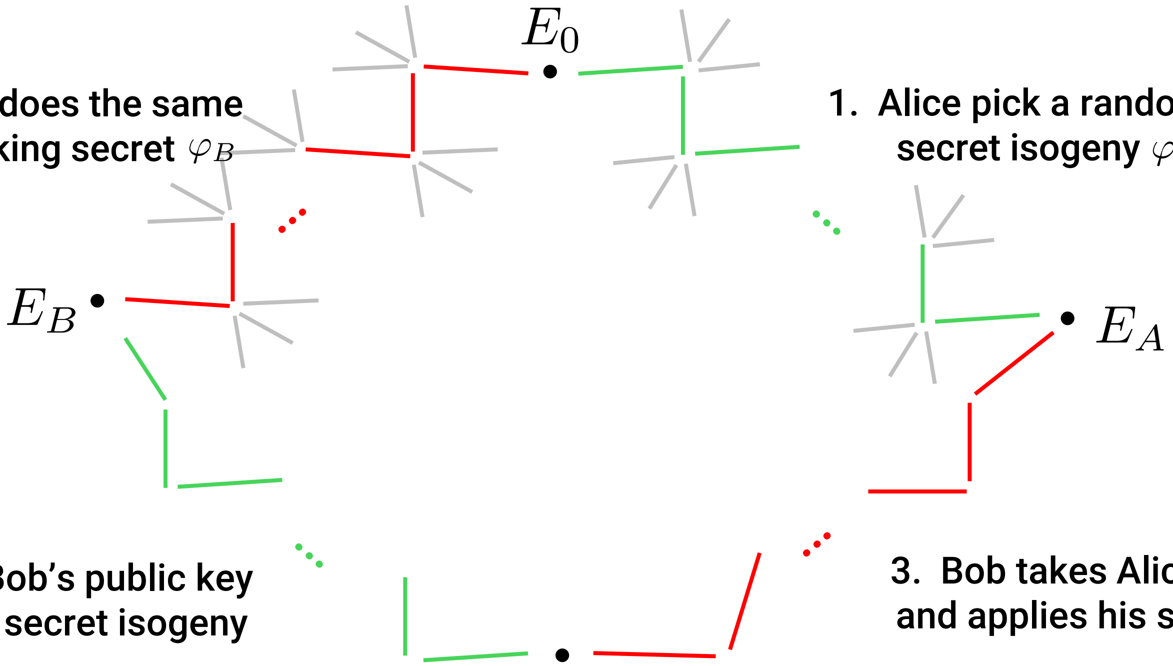

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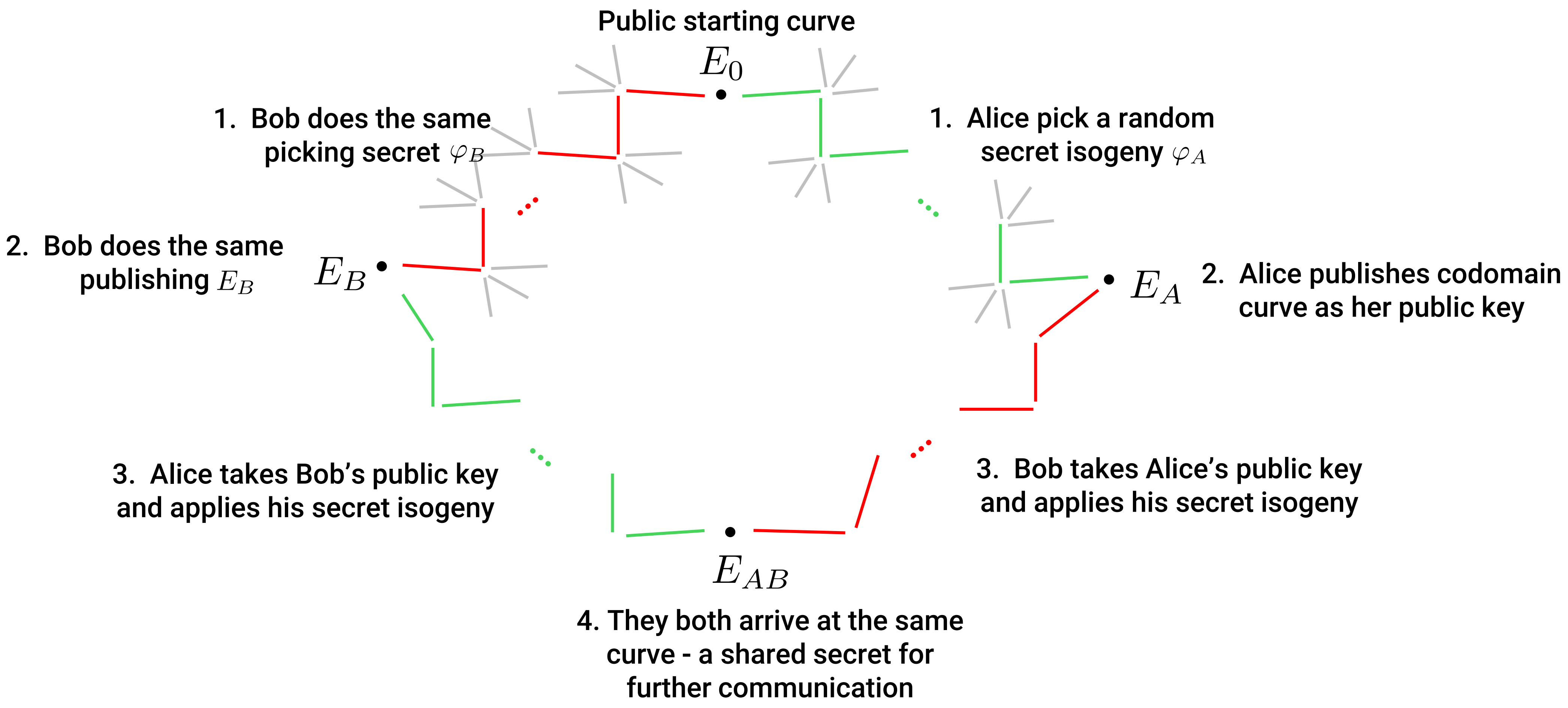

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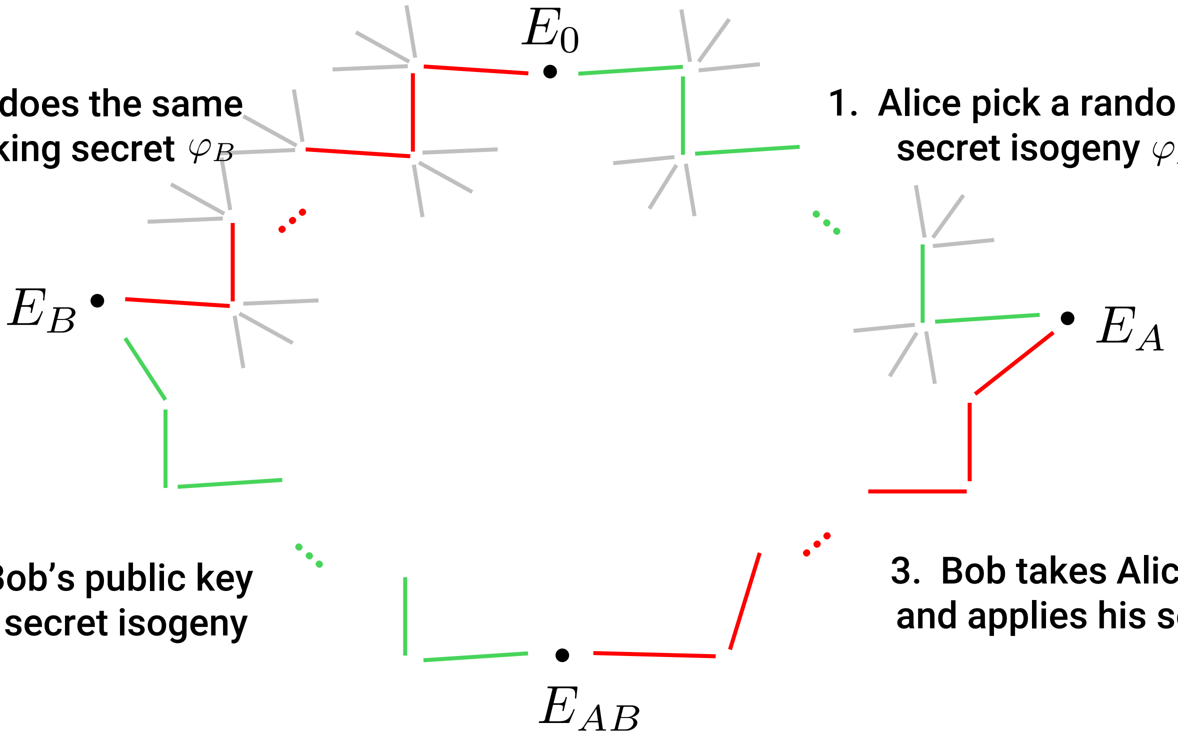

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4. They both arrive at the same curve - a shared secret for further communication

(Detail Omitted - In step 3, how does Alice "evaluate" their isogeny $\varphi_A : E_0 \to E_A$ from a different curve E_B in a commutative way?)

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Any quaternion $\alpha \in B$ with integral norm and trace $\mathrm{nrd}(\alpha), \mathrm{Tr}(\alpha) \in \mathbb{Z}$ is a **quaternion integer**.

Orders and Ideals

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An **integral lattice** is the linear span of 4 quaternion integers over $\mathbb{Z}$.

$L = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ for generators $e_0, e_1, e_2, e_3 \in B$ with $\mathrm{nrd}(e_i), \mathrm{Tr}(e_i) \in \mathbb{Z}$

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And each ideal “connects” two maximal orders $I = N \cdot \mathcal{O}_1\mathcal{O}_2$.

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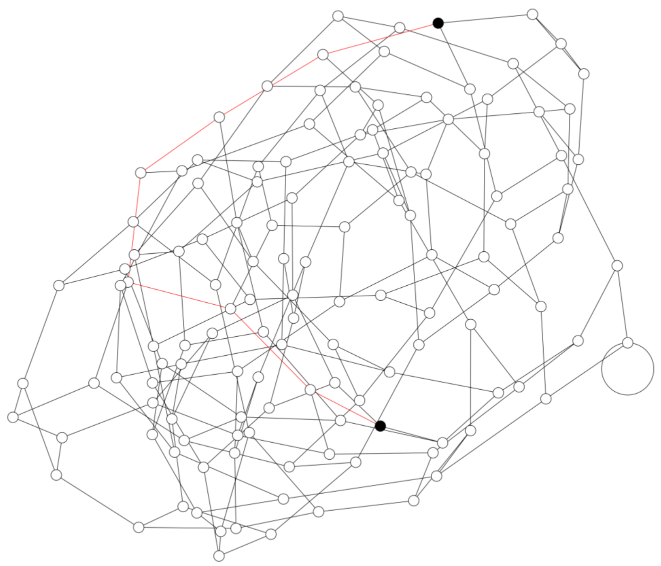

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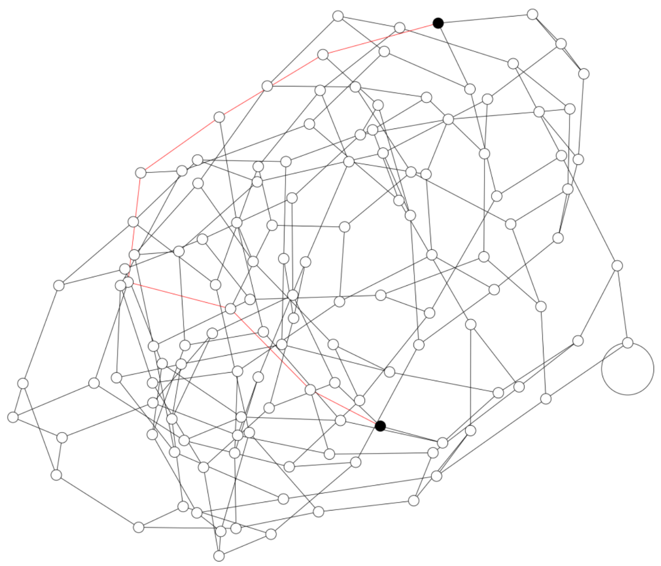

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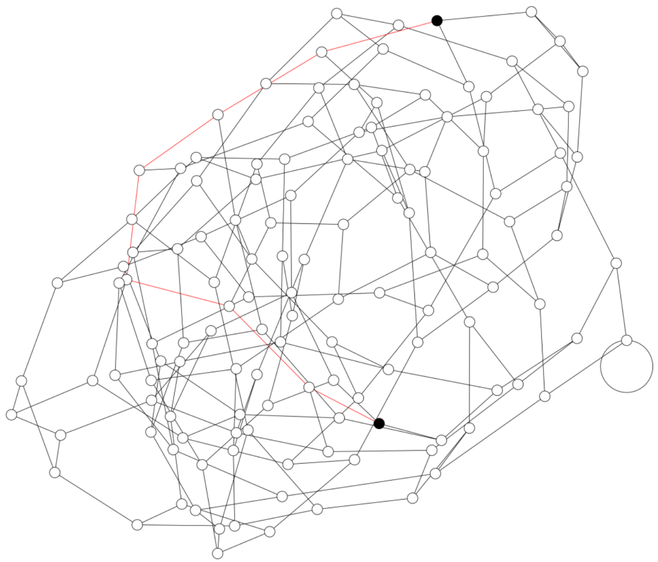

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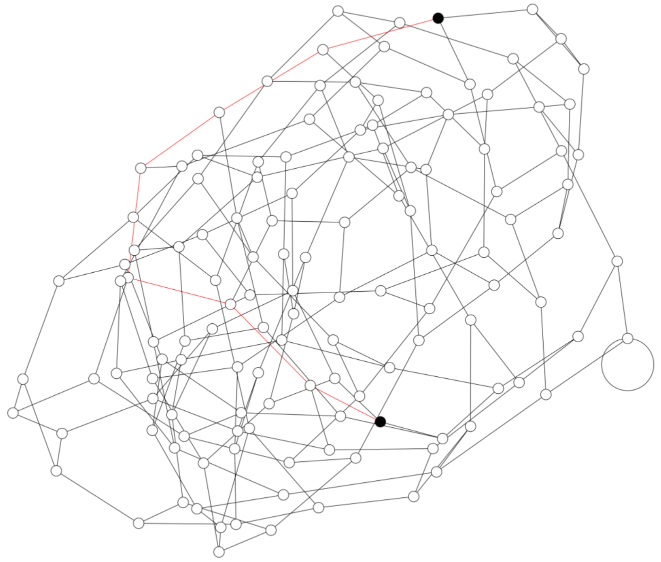

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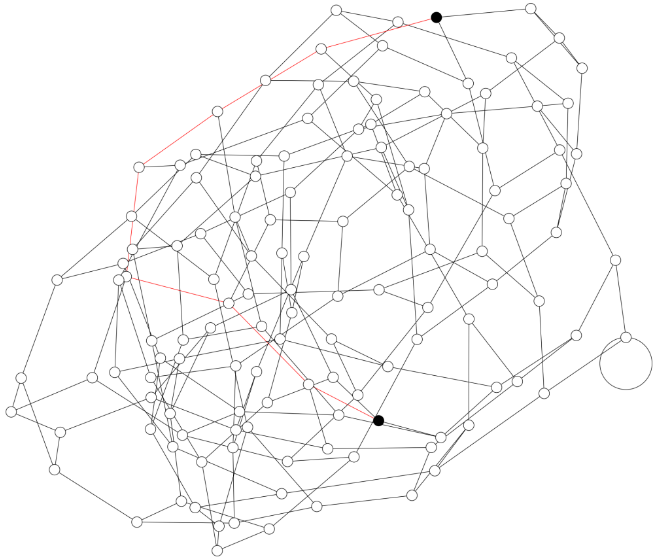

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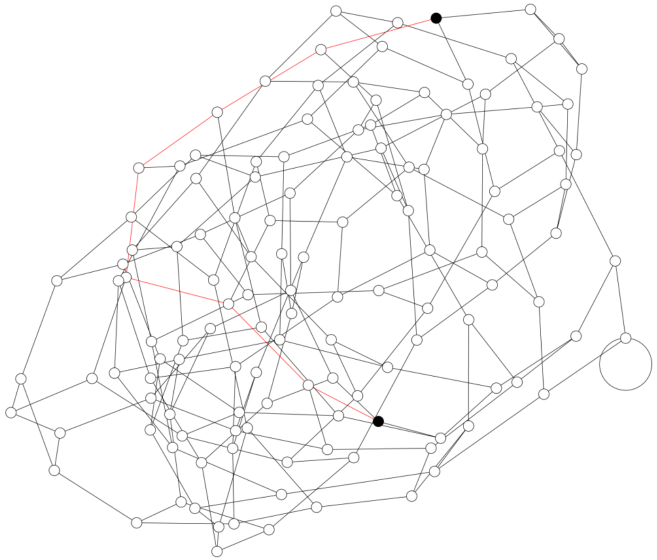

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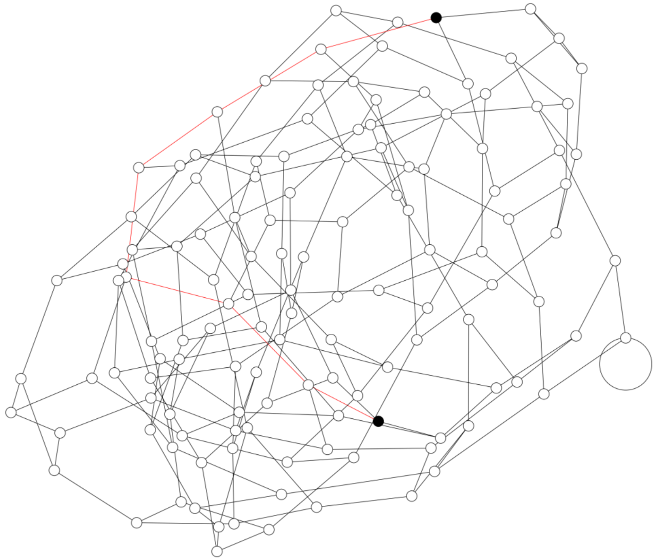

This is the **Deuring Correspondence** relating the two worlds of isogenies and quaternions.

What I'm trying to achieve…

Finding best algorithms to solve the Quaternion Embedding Problem.

Given a maximal order $\mathcal{O}$ find an element $\alpha \in \mathcal{O}$ of prescribed trace t and norm d .

Hardness of this problem gives arguments for the hardness of isogeny problems in general.

Finding shortest norm q^n ideal paths connecting two maximal orders $\mathcal{O}_1$ and $\mathcal{O}_2$.

This would result in major speedups to digital signature scheme SQISign and give better estimates to aid security analysis.

Fast constant-time sampling of random ideals of a given norm.

Giving further improvements to SQISign.

Thanks!